

Mapping cones and separable states

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Abstract

We study mapping cones and their dual cones of positive maps of the $n \times n$ matrices into itself. For a natural class of cones there is a close relationship between maps in the cone, super-positive maps, and separable states. In particular the composition of a map from the cone with a map in the dual cone is super-positive, and so the natural state it defines is separable.

1 Introduction

One approach to the study of positive maps between operator algebras is via mapping cones. They were introduced in [5] and are closed convex cones of positive maps closed under composition with completely positive maps. They are especially useful in finite dimensions as they yield much information on positive maps with given positivity properties, see e.g. [6]. In the present paper we shall study mapping cones which are closed under composition with all positive maps. Then we obtain characterizations of super-positive maps, also called entanglement breaking maps, and their relationship to separable operators and states.

Three classes of maps will be central in the present paper. Let M_n denote the complex $n \times n$ matrices, and let $\phi: M_n \rightarrow M_n$ be a linear map. Then ϕ is *positive*, written $\phi \geq 0$, if $\phi(a) \geq 0$ whenever $a \in M_n^+$, the positive matrices in M_n . ϕ is *completely positive* if $\iota \otimes \phi \geq 0$, where ι is the identity map on M_n . Then ϕ is of the form $\phi = \sum_i \text{Ad} V_i$, finite sum, where $V_i \in M_n$, and $\text{Ad} V(a) = V^* a V$, see [6] Thm. 4.1.8. ϕ is *super-positive* if it is of the form $\sum_i a_i \omega_i$, where $a_i \in M_n^+$, and ω_i is a state on M_n . The study of positive maps ϕ is equivalent to the study of its *Choi matrix* denoted by C_ϕ and defined as follows.

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) \in M_n \otimes M_n,$$

where e_{ij} , $1 \leq i, j \leq n$, is a complete set of matrix units for M_n . If Tr is the usual trace on $M_n \otimes M_n$ then we have, see [6], Ch. 4. $\phi \geq 0$ iff $\text{Tr}(C_\phi a \otimes b) \geq 0$ for all $a, b \in M_n^+$, and this holds iff $\text{Tr}(C_\phi C_\psi) \geq 0$ for all super-positive ψ .

ϕ is completely positive iff $C_\phi \geq 0$. ϕ is super-positive iff $\text{Tr}(C_\phi C_\psi) \geq 0$ for all positive maps ψ iff C_ϕ is separable, i.e.

$$C_\phi = \sum a_i \otimes b_i, a_i, b_i \in M_n^+.$$

We shall use the notation

P is the cone of all positive maps of M_n into itself.

CP is the cone of all completely positive maps.

SP is the cone of all super-positive maps.

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2 Mapping cones and their dual cones.

As above P denotes the set of positive maps of M_n into itself. A *mapping cone* is a closed convex cone $J \subset P$ such that $\phi \in J$ implies $\alpha \circ \phi \circ \beta \in J$ whenever $\alpha, \beta \in CP$. We always assume J is *symmetric*, i.e. $\phi \in J$ implies $\phi^* \in J$, and $\phi^t \in J$, where ϕ^* and ϕ^t are defined by $\text{Tr}(\phi(a)b) = \text{Tr}(a\phi^*(b))$, $\phi^t(a) = \phi(a^t)^t$, $\forall a, b \in M_n$, where t denotes the transpose.

J is an *invariant mapping cone* if $\phi \in J$ implies $\alpha \circ \phi \circ \beta \in J$ for all $\alpha, \beta \in P$

The *dual cone* of J is the cone

$$J^o = \{\phi \in P : \text{Tr}(C_\phi C_\psi) \geq 0, \forall \psi \in J\}.$$

Then $J^{oo} = J$, see [6], Section 6.2, and J^o is a mapping cone which is invariant if J is. The definitions of P, CP, SP in the introduction can be restated as

$$\begin{aligned} \phi \in P &\Leftrightarrow \phi \in SP^o. \\ \phi \in CP &\Leftrightarrow \phi \in CP^o. \\ \phi \in SP &\Leftrightarrow \phi \in P^o. \end{aligned}$$

The next result with a different proof is the same as [6], Lem.5.1.5.

Lemma 1 *SP is an invariant mapping cone, and is also a minimal mapping cone.*

Proof. SP is clearly an invariant mapping cone since if ω is a state and $\phi \in P$ then $\omega \circ \phi$ is a scalar multiple of a state, so belongs to SP , and $\phi \circ \omega(a) = \omega(a)\phi(1)$ is also in SP . If $\psi \in SP$ and $\phi \in P$, then $\text{Tr}(C_\phi C_\psi) \geq 0$. In particular this holds for $\phi \in J^o$, hence $\psi \in J^{oo} = J$. Thus $SP \subset J$, completing the proof.

Note that SP is not a hereditary cone, because the trace Tr_n on M_n belongs to SP , and by [6], Thm. 7.5.4, if $\phi \in P$ then $\psi = \phi(1)\text{Tr}_n + \phi \in SP$, and so $\phi = \psi - \phi(1)\text{Tr}_n$ is the difference of two maps in SP , hence $\phi \leq \psi$ even though ϕ need not belong to SP .

The following theorem is central for our development of the theory to follow. It is a slight variation of Theorems 6.6 and 7.1.1 in [6]. Recall that if $\phi \in P$ then $\tilde{\phi}$ is the linear functional on $M_n \otimes M_n$ defined by

$$\tilde{\phi}(a \otimes b) = \text{Tr}(\phi(a)b^t).$$

By [6], Lem. 4.2.3, C_ϕ^t is the density matrix for $\tilde{\phi}$. Thus $\tilde{\phi}$ is positive iff ϕ is completely positive. By [6] Prop. 5.1.4 $\tilde{\phi}$ is separable iff C_ϕ is separable.

Theorem 2 *Let J be a mapping cone and $\phi \in P$. Then the following conditions are equivalent.*

- (i) $\phi \in J^\circ$.
- (ii) $\phi \circ \alpha \in CP \forall \alpha \in J$.
- (iii) $\alpha \circ \phi \in CP \forall \alpha \in J$.
- (iv) $\tilde{\phi} \circ (\iota \otimes \alpha) \geq 0 \forall \alpha \in J$.
- (v) $(\iota \otimes \alpha)(C_\phi) \geq 0 \forall \alpha \in J$.

Proof. The only part of the theorem which is not contained in [6], Thm. 6.1.6 is the equivalence of (ii) with the others. But we have by symmetry of CP that

$$\phi \circ \alpha \in CP \Leftrightarrow (\phi \circ \alpha)^* = \alpha^* \circ \phi^* \in CP.$$

Since J is symmetric, as we have assumed for all our mapping cones, $\phi \circ \alpha \in CP$ iff $\alpha \circ \phi^* \in CP$ for all $\alpha \in J$, hence by the equivalence (i) \Leftrightarrow (iii), iff $\phi^* \in J^\circ$. But J° is symmetric, so $\phi^* \in J^\circ$ iff $\phi \in J^\circ$, hence we have (i) \Leftrightarrow (ii), completing the proof.

If J is an invariant mapping cone and $\phi \in P$, conditions (ii) and (iii) can be sharpened.

Corollary 3 *Let J be an invariant mapping cone and $\phi \in P$. Then the conditions in Theorem 2 are equivalent to*

- (vi) $\phi \circ \alpha \in SP \forall \alpha \in J$.
- (vii) $\alpha \circ \phi \in SP \forall \alpha \in J$.

Proof. Let $\alpha, \psi \in P$. Then we have for $\alpha \in J$

$$\text{Tr}(C_{\alpha \circ \phi} C_\psi) = \text{Tr}(\iota \otimes \alpha(C_\phi) C_\psi) = \text{Tr}(C_\phi C_{\alpha^* \circ \psi}).$$

Hence

$$\text{Tr}(C_{\alpha \circ \phi} C_\psi) \geq 0 \forall \psi \Leftrightarrow \phi \in J^\circ,$$

as the maps $\alpha^* \circ \psi$ include all maps in J . Thus $\phi \in J^\circ$ iff $\alpha \circ \phi \in SP$ for all $\alpha \in J$. As in the proof of the equivalence (ii) \Leftrightarrow (iii) in Theorem 2 we show (vi) \Leftrightarrow (vii). The proof is complete.

If V is a finite dimensional Hilbert space, and J and K are closed convex cones in V , then the dual cone $(J \cap K)^\circ$ of their intersection is the closure of the sum $J^\circ + K^\circ$ of their dual cones, see [1]. From this we get the following corollary, which is included for completeness.

Corollary 4 *Let J and K be mapping cones in P . Then $(J \cap K)^o = J^o + K^o$.*

Proof. By the above and finite dimensionality it suffices to show that $J^o + K^o$ is closed in the norm topology. So let $(\phi_i = \alpha_i + \beta_i)$ be a sequence converging to ϕ , where $\alpha_i \in J^o, \beta_i \in K^o$. We can assume $\|\phi_i\| \leq \|\phi\|$ for all i and the same for α_i and β_i . Since the dimension is finite, by compactness of the unit ball in J^o there is a subsequence of (α_i) which converges to a map $\alpha \in J^o$ and a subsequence (β_j) of (β_i) which converges to $\beta \in J^o$. Thus

$$\phi_j = \alpha_j + \beta_j \longrightarrow \phi = \alpha + \beta \in J^o + K^o,$$

proving the corollary.

Remark. In the sequel we shall see that maps of the form Ade defined by $Ade(a) = eae$ are of central importance. If $a \in M_n$ with $\text{range}(a) = e$, then for J a mapping cone $Ade \in J$ iff $Ada \in J$. Indeed, we have $a = ae$, so $Ada = Ada \circ Ade \in J$ if $Ade \in J$, and if $Ada \in J$, since $\text{range}(aa^*) = e$ we can assume $a \geq 0$. Let a^{-1} be the inverse of a in eM_ne , so $aa^{-1} = e$. $Ade = Ada \circ Ada^{-1} \in J$, proving the converse.

Theorem 5 *Let J be an invariant mapping cone. Let*

$$k = \max\{\text{rank}(e) : Ade \in J\}.$$

Assume $k \geq 2$. Then we have

- (i) *If $\phi \in J^o$ and $\text{rank}(\text{range}(\phi)) \leq k$, then $\phi \in SP$.*
- (ii) *If e is a projection in M_n with $\text{rank}(e) \leq k$, then $Ade \in J$.*

Proof. Let $\phi \in J^o$ be as in (i). By assumption there exists a projection e of rank k such that $Ade \in J$. Let f be the range projection of ϕ , so by assumption $\text{rank}(f) \leq \text{rank}(e)$. Thus there exists a partial isometry $v \in M_n$ such that $f = v^*ev$. Hence

$$Adf = Adv \circ Ade \circ Adv^* \in J.$$

Thus $\phi = Adf \circ \phi \in J$, hence ϕ is the composition of a map in J with a map in J^o , so $\phi \in SP$ by Corollary 3, proving (i). To show (ii) let f be a projection of rank k such that $f \geq e$. By the above argument $Adf \in J$ hence $Ade = Adf \circ Ade \in J$.

Corollary 6 *Let $n=2$ and J be an invariant mapping cone in P . Then either $J = P$ or $J = SP$.*

Proof. Let k be as in Theorem 5. Since every map in P is decomposable, see [6], Thm. 6.3.1, each map is a linear combination of maps of the form Adv or $Adv \circ t$, where $v \in M_2$, and t denotes the transpose. If $k = 1$ it thus follows that all maps in J are super-positive, hence in SP . If $k = 2$ then the identity map belongs to J , so $J = P$, completing the proof of the corollary.

We denote by D_2 the invariant mapping cone generated by Ade for the rank 2 projections in M_n , or by the remark before Theorem 5, matrices a of rank 2. By the argument in the proof of Theorem 5 D_2 is the invariant mapping cone generated by a single map Ade with e of rank 2. By Theorem 5, if $n > 2$, and Corollary 6 if $n = 2$, we have

Corollary 7 *If e is a projection with $Ade \in D_2^o$, and $\text{rank}(e) \leq 2$ then $\text{rank}(e) = 1$.*

Theorem 8 *Let J be an invariant mapping cone not contained in D_2^o . Then J contains D_2 .*

Proof. Let $\phi \in J$, but not in D_2^o . Then in particular ϕ does not belong to SP . We assert that there exist $\alpha, \beta, \gamma \in P$ and projections e, f of rank 2 such that

$$Ade \circ \alpha \circ \phi \circ \beta \circ Adf \circ \gamma \notin SP.$$

If the assertion is false, i.e. the above map belongs to SP for all $\alpha, \beta, \gamma, e, f$, then, since the maps $\beta \circ Adf \circ \gamma$ generate D_2 , it follows from Corollary 3 that

$$Ade \circ \alpha \circ \phi \in D_2^o \quad \forall e, \alpha.$$

Again, since $Ade \in D_2$, by Corollary 3,

$$Ade \circ \alpha \circ \phi = Ade \circ (Ade \circ \alpha \circ \phi) \in SP.$$

It follows that from the last equation

$$(\eta \circ Ade \circ \alpha) \circ \phi \in SP \quad \forall \eta, \alpha \in P.$$

Since the maps $\eta \circ Ade \circ \alpha$ generate D_2 , it follows again from Corollary 3 that $\phi \in D_2^o$, contrary to assertion at the beginning of the proof. Thus the assertion follows. We can therefore find projections e and f of rank 2 and $\alpha, \beta \in P$ such that

$$Ade \circ \alpha \circ \phi \circ \beta \circ Adf \notin SP.$$

As in the proof of Theorem 5 (i) there exists a partial isometry v such that $Adf = Adv \circ Ade \circ Adv^*$, so we can replace f by e , and thus find e of rank 2 such that

$$Ade \circ (\alpha \circ \phi \circ \beta) \circ Ade \notin SP.$$

Let

$$J_2 = \{Ade \circ \psi \circ Ade : \psi \in J\}.$$

Since $\alpha \circ \phi \circ \beta \in J$ it follows by the above that $J_2 \not\subseteq SP$. J_2 is an invariant mapping cone inside the set

$$\{Ade \circ \psi \circ Ade : \psi \in P\},$$

which is linearly isomorphic to the positive maps of M_2 into itself. By Corollary 6, since $J_2 \not\subseteq SP$

$$J_2 = \{Ade \circ \psi \circ Ade : \psi \in P\}.$$

But then $Ade \in J_2$. By assumption $\phi \in J$, and $J_2 \subset J$, so $Ade \in J$. Since Ade generates D_2 as an invariant mapping cone, we thus conclude that $D_2 \subset J$. The proof is complete

A map $\phi \in P$ is called *extremal* if whenever $\psi \in P$ and $\phi \geq \psi$, then $\psi = \lambda\phi$ for some $0 \leq \lambda \leq 1$. This can be formulated as, if $\phi = \sum_{i=1}^k \psi_i$ is a sum of positive maps ψ_i , then $k = 1$, so $\phi = \psi_1$. It is shown in [6] Prop. 3.1.3. that each map of the form $\phi = Adv$ is extremal.

Proposition 9 . Let $\phi \in P$ be an extremal map, and let J be the invariant mapping cone generated by ϕ . Then we have:

- (i) If $\phi \notin D_2 \vee D_2^o$ then $J \supsetneq D_2$, and $J \not\subseteq D_2^o$.
- (ii) If ϕ is unital then $J \not\subseteq P$ unless ϕ is either an automorphism or anti-automorphism.

Proof. Ad (i). Clearly $J \not\subseteq D_2^o$, hence by Theorem 8 $J \supseteq D_2$. Since $\phi \notin D_2$, $J \not\subseteq D_2$. Ad(ii). If $J = P$ then the identity map $\iota \in J$. Since ι is an extremal map it must be of the form $\iota = \alpha \circ \phi \circ \beta$ with $\alpha, \beta \in P$. Furthermore, α and β must be extremal, since otherwise ι would be a sum of several positive maps. We thus have β is invertible with inverse $\alpha \circ \phi$. Thus β is an order-isomorphism of M_n onto itself. Similarly α is an order-isomorphism, so that ϕ is the same. Since ϕ is unital, ϕ is either an automorphism or an anti-automorphism, see e.g. [6] Thm.2.1.3. The proof is complete.

3 PPT-states

PPT-states were introduced by Peres [4] in 1996 and are states ρ on $M_n \otimes M_n$ such that $\rho \circ (\iota \otimes t)$ is a state, where t as before is the transpose on M_n . For a while it was believed that they were separable, but a counter example was exhibited by P. Horodecki [3] in 1997.

We call a map $\phi \in P$ for a *PPT-map* if the corresponding linear functional $\tilde{\phi}$ is a PPT-state. The following consequences can be read out of [6], Section 7.2.

Proposition 10 Let $\phi \in P$. Then the following conditions are equivalent.

- (i) $\phi \in CP \cap coCP$.
- (ii) $\phi \circ \alpha \in CP \forall \alpha \in CP \vee coCP$.
- (iii) ϕ is a PPT-map.

Recall that map $\phi \in CP \vee coCP$ is decomposable, so it is of the form $\phi = \alpha + \beta$ with $\alpha \in CP$, $\beta \in coCP$, see Corollary 4. The next result is a dual version of Proposition 10. In (ii) we used that $CP \cap coCP$ is a mapping cone with dual cone $CP \vee coCP$.

Proposition 11 . Let $\phi \in P$. Then the following conditions are equivalent.

- (i) $\phi \in CP \vee coCP$.
- (ii) $\phi \circ \alpha \in CP \cap coCP$, $\forall \alpha \in CP \vee coCP$.
- (iii) $\tilde{\rho}(C_\phi) \geq 0$, \forall PPT-maps ρ

If $n=3$, then D_2 consists of decomposable maps. For completeness we include a proof of this fact.

Lemma 12 Let $n=3$. Then D_2 consists of decomposable maps, hence $D_2 \subset CP \vee coCP$.

Proof. Let $\phi \in D_2$. Then ϕ is a sum of maps $\alpha \circ Ade \circ \beta$ with e a projection of rank ≤ 2 , and $\alpha, \beta \in P$. Each map $\alpha \circ Ade$ can be identified with a map $M_2 \rightarrow M_3$, and the map $Ade \circ \beta$ with a map $M_3 \rightarrow M_2$. By a result of Woronowicz [7], both maps are decomposable, hence so is $\alpha \circ Ade \circ \beta$, and therefore ϕ is decomposable, so belongs to $CP \vee coCP$, proving the lemma.

Proposition 13 . *Let $n=3$ and $\phi \in P$ be a PPT-map. Then we have:*

- (i) $\phi \in D_2^o$.
- (ii) $\phi \circ \alpha \in SP$ and $\alpha \circ \phi \in SP \forall \alpha \in D_2$.
- (iii) *If $range(\phi)$ or $supp(\phi)$ is a projection of rank 2, then $\phi \in SP$*

Proof. By Proposition 10 and Lemma 12 $\phi \in CP \cap coCP = (CP \vee coCP)^o \subset D_2^o$, proving (i). Thus by Corollary 3, $\phi \circ \alpha \in SP$, and $\alpha \circ \phi \in SP$ for all $\alpha \in D_2$, proving (ii). To show (iii) note that under the assumptions on range and support of ϕ , $Ad(range(\phi))$ or $Ad(supp(\phi))$ belongs to D_2 . Hence, either $\phi = Ad(range(\phi)) \circ \phi \in SP$ by (ii), or similarly for $\phi = \phi \circ Ad(supp(\phi)) \in SP$, so $\phi \in SP$ completing the proof.

In [2] it was shown that all PPT-states on $M_2 \otimes M_2$ and $M_3 \otimes M_2$ are separable. These results follow easily from the above results. Indeed if $n=2$ then by Proposition 10 and Corollary 6 we get

$$\phi \in CP \cap coCP = (CP \vee coCP)^o \subset D_2^o = SP.$$

When $n=3$, and $e = range(\phi)$ is of rank 2 then, since $eM_3e \simeq M_2$, we can consider ϕ as a PPT-map from M_3 to M_2 . By Proposition 13(iii) $\phi \in SP$. To translate this to states recall that C_ϕ^t is the density matrix for $\tilde{\phi}$, and $\tilde{\phi}$ is separable iff $\phi \in SP$.

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